Modeling walker synchronization on the Millennium Bridge

Bruno Eckhardt
Fachbereich Physik, Philipps-Universität Marburg, D-35032 Marburg, Germany and IREAP, IPST, and Burgers Program, University of Maryland, College Park, Maryland 20742, USA

Edward Ott
IREAP, Department of Physics and Department of Electrical and Computer Engineering, University of Maryland, College Park, Maryland 20742, USA

Steven H. Strogatz and Daniel M. Abrams
Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853-1503, USA

Allan McRobie
Department of Engineering, Cambridge University, Cambridge CB2 1PZ, United Kingdom
(Received 11 June 2006; published 12 February 2007)

On its opening day the London Millennium footbridge experienced unexpected large amplitude wobbling subsequent to the migration of pedestrians onto the bridge. Modeling the stepping of the pedestrians on the bridge as phase oscillators, we obtain a model for the combined dynamics of people and the bridge that is analytically tractable. It provides predictions for the phase dynamics of individual walkers and for the critical number of people for the onset of oscillations. Numerical simulations and analytical estimates reproduce the linear relation between pedestrian force and bridge velocity as observed in experiments. They allow prediction of the amplitude of bridge motion, the rate of relaxation to the synchronized state and the magnitude of the fluctuations due to a finite number of people.

DOI: 10.1103/PhysRevE.75.021110 PACS number(s): 05.90.+m, 05.45.Xt

I. INTRODUCTION

On its opening day on June 10, 2000, the Millennium footbridge over the River Thames in London attracted thousands of spectators. Many on the bridge, however, were scared by a quite noticeable and disturbing lateral vibration, reaching amplitudes of several centimeters and causing people to reach to hold onto the handrails for stability. A partial solution was found for the second day, when access to the bridge was limited, but eventually the bridge was closed until a cure to the problem had been implemented [1–3]. Investigations by the engineering team involved in the bridge design quickly eliminated several possible causes. It could not be attributed to differences in mechanical behavior between the design studies and the actual bridge, nor to vortex shedding in the presence of wind, the reason for the collapse of the Tacoma Narrows bridge in 1940 [4], nor to deviations from established procedures and guidelines. Various models have been proposed to describe the dynamics, [1–3,5–7] but they do not account for the transition to synchronized dynamics. Perhaps closest to the ideas presented here comes the letter by Josephson [8], in which he points out similarities to the synergetics of Haken [9].

The key to an understanding of the problem came from controlled studies with variable numbers of walkers on the bridge. It was found that the oscillations set in quickly once the number of walkers on the bridge exceeded a critical number and that, when this occurred, many started to move in synchronized step. During these investigations [3,10] several previous examples on other footbridges were identified. The model for the synchronized walking presented here (i) explains the transition to synchronization, (ii) predicts a relation between the critical number of walkers and a combination of characteristics of the bridge and human motion, and (iii) explains the observed proportionality between the bridge velocity and the force exerted by pedestrians. It provides a basis on which the findings for the Millennium bridge can be generalized and extended to other bridges. The model can also serve as a guide for further behavioral and experimental studies.

At the heart of the theoretical analysis is a suitable model for the motion of pedestrians and their interaction with the bridge. In Sec. II we discuss the assumptions and mathematical simplifications that lead to the explicit functional form used in the numerical studies. Within a linear stability analysis we derive in Sec. III the critical parameters for the onset of the bridge oscillation and discuss the dependence on damping, frequency ratios and pedestrian parameters. In Sec. IV we present analytical and numerical results on the nonlinear behavior. The determination of the various parameters and the comparison to observations on the London Millennium Bridge is discussed in Sec. V. Some concluding remarks are given in Sec. VI.

Finally, we note that a brief version of this work has been presented in Ref. [11]. In that paper a closely related, but slightly different, model was employed, and we discuss its relation to the model used in the present paper in Sec. VI.

II. MODELING

The bridge motion can be expanded in normal modes so that the lateral oscillations can be described as damped harmonic oscillators forced by the motion of the pedestrians,

$$M\ddot{y}+2M\zeta\Omega\dot{y}+M\Omega^2 y=\sum_{i=1}^{N}\hat{f}_i(t), \tag{1}$$

where $y(t)$ is the modal bridge displacement, Ω its (angular) eigenfrequency, M the equivalent modal mass, ζ the (dimensionless) damping constant, and the dots on y denote time derivatives. The lateral modal force exerted on the bridge by pedestrian i is $\hat{f}_i(t)$, where $i=1,2,\ldots,N$, with N the number of pedestrians on the bridge.

In order to self-consistently model the dynamics of the bridge-pedestrian system, we must incorporate the dynamics of the different responses of individual pedestrians as they adjust their stepping under the influence of the bridge motion. A fundamental difficulty is that there does not currently exist a well-developed, generally accepted, physiological model of human walking dynamics and its response to external inputs. In the absence of such knowledge, we seek to construct a general model using as few assumptions as possible.

For this purpose, it is useful to restrict consideration to the construction of a model of the response of walkers to small bridge displacements, and to assume that the walker response is approximately linear in the bridge displacement. This assumption will be further discussed at the end of this section. In addition, we employ the following hypotheses.

(i) We assume that the effect results solely from an interaction between the bridge and the walkers and not from visual or other communication between the walkers: people-people interactions are not included in the model.

(ii) The only significant bridge variable sensed by the walkers is that due to the side-to-side force felt by the walkers in the moving frame of the bridge [i.e., the walkers directly sense only the side-to-side bridge acceleration $\ddot{y}(t)$].

(iii) The dynamics of a walker's response to small amplitude bridge motion is describable within the phase oscillator framework (described below).

Regarding hypothesis (ii), it might be argued that a walker could directly sense the bridge position $y(t)$ from some visual input, e.g., the motion of other walkers or the bridges displacement relative to the supports. This, however, requires that the view of the bridge is not blocked by other walkers. We expect that visual input from observing other walkers is rather indirect and that orientation on far away fixed points (such as St. Pauls cathedral) is, because of the small parallax, much less significance than the acceleration.

Regarding hypothesis (iii), in the absence of bridge motion, the stepping of walker i is taken to be periodic with an angular frequency ω_i. Thus, in the absence of bridge motion, we can write

$$\hat{f}_i(t)=f_i(\theta_i(t)), \tag{2}$$

$$\dot{\theta}_i=\omega_i. \tag{3}$$

What is meant by hypothesis (iii) is that, in the presence of bridge motion, we retain Eq. (2) and model the effect of the bridge on walker i by an additional term in (3) affecting the evolution of the phase of walker i,

$$\dot{\theta}_i=\omega_i-F_i(t). \tag{4}$$

This type of description is motivated by the observation that, when there is no bridge motion, the terms in the sum on the right-hand side of (1) tend to cancel for a large number of walkers due to the phase incoherence of the walkers' stepping. Note that this applies even though the amplitude of the lateral force of *individual* walkers is nonzero. Thus, in the linear stage of instability, only perturbations of the walker phases are important. (We emphasize that this cannot be concluded to apply in the strongly nonlinear case of large bridge oscillations.) Note also that the above justification of the phase oscillator description is essentially the same as that used by Kuramoto [12] in formulating his well-known coupled oscillator model [13].

The immediate problem is now seen to be that of modeling the quantity $F_i(t)$. Most generally, $F_i(t)$ depends on t through the values of the functions $\ddot{y}(t')$ and $\theta_i(t')$ for all time $t'\leq t$. That is, for fixed t, $F_i(t)$ is a functional of the past history of $\ddot{y}$ and θ_i. In addition, as previously stated, motivated by our initial restriction to the case of small $\ddot{y}$, we take $F_i(t)$ to be linear in $\ddot{y}$,

$$F_i(t)=\int_{-\infty}^{t}L_i(t,t';\theta(t'')|t''\leq t)\ddot{y}(t')dt', \tag{5}$$

where L_i is a function of t and t' and a functional of $\theta_i(t'')$ for $t''\leq t$.

We now make the further assumption that $F_i(t)$ depends only on the value of θ at the current time t and not on its past history. This assumption greatly facilitates analysis, and justification for it is given at the end of this section. We can then rewrite (5) as

$$F_i(t)=\int_{-\infty}^{t}\hat{L}_i(t,t',\theta_i(t))\ddot{y}(t')dt', \tag{6}$$

where $\hat{L}_i$ is a *function* of its three arguments. Since we require (6) to be independent of the choice labeling the time instant $t=0$, we require that (6) be invariant to the transformation: $F_i(t)\to F_i(t+T)$, $\ddot{y}(t)\to\ddot{y}(t+T)$, $\theta_i(t)\to\theta_i(t+T)$. Inserting this transformation in (6), and defining $\tilde{t}=t+T$ and $\tilde{t}'=t'+T$, we recover (6), but with $\hat{L}_i(t,t',\theta_i)$ replaced by $\hat{L}_i(\tilde{t}-T,\tilde{t}'-T,\theta_i)$, which, by the presumed invariance to the labeling of the zero of time, is independent of T. Hence $\hat{L}_i$ must have the form

$$\hat{L}_i(t,t',\theta_i)=\hat{\mathcal{L}}_i(t-t',\theta_i). \tag{7}$$

Expanding $\hat{\mathcal{L}}_i$ in Fourier series, we have

$$\hat{\mathcal{L}}_i(\tau,\theta)=\sum_{n=-\infty}^{+\infty}\hat{\mathcal{L}}_i^{(n)}(\tau)\exp(in\theta). \tag{8}$$

Thus (6) becomes

$$F_i(t) = \sum_{n=-\infty}^{+\infty} \exp[in\theta_i(t)]\mathcal{L}_i^{(n)}[\ddot{y}(t)], \qquad (9)$$

where $\mathcal{L}_i^{(n)}$ denotes the linear operator defined by

$$\mathcal{L}_i^{(n)}[h(t)] = \int_{-\infty}^{t} \hat{\mathcal{L}}_i^{(n)}(t-t')h(t')dt', \qquad (10)$$

and, since $\hat{\mathcal{L}}_i$ is real,

$$\hat{\mathcal{L}}_i^{(n)} = (\hat{\mathcal{L}}_i^{(-n)})^*, \qquad (11)$$

where * denotes the complex conjugate.

To summarize, our model so far reduces to Eqs. (1), (2), (4), (9), and (10). A linear stability analysis of these equations is given in Sec. III where three important additional facts are invoked to further simplify the model for walkers on the Millennium Bridge. These facts are the following.

(1) The dimensionless damping constant ζ in Eq. (1) is small [1,2].

(2) The unperturbed walker frequencies ω_i have been measured for large populations of walkers [14], and the distribution of walker frequencies has been found to have a spread σ (square root of the variance of ω_i) that is small compared to the population averaged frequency, μ.

(3) For the Millennium Bridge, Ω and μ are very close in value [1,2].

Regarding (3) we note that this applies well to observations of the north span of the bridge. Thus our model does not seek to address oscillations at the lower frequency sometimes observed on the bridge's central span. We focus on the oscillation for the north span because it is the only mode for which data on the critical number of pedestrians has been presented [1,2].

Taking the small quantities ξ, σ/μ, and $|\Omega-\mu|/\Omega$ to be of the same order, then makes possible a perturbation expansion. Important conclusions from this analysis (Sec. III) are that, to lowest order in the perturbation expansion, the following apply to the linear analysis:

(a) only the $n=\pm 1$ terms in (9) contribute;

(b) the operator $\mathcal{L}_i^{(\pm 1)}$ is approximately constant,

$$\mathcal{L}_i^{(+1)}[h(t)] = C_i h(t), \mathcal{L}_i^{(-1)}[h(t)] = C_i^* h(t), \qquad (12)$$

where C_i is a complex number characterizing walker i;

(c) only the fundamental harmonic of $f_i(\theta_i)$ [appearing in (2)] contributes [i.e., in the Fourier series $f_i(\theta) = \Sigma_m f_i^{(m)} \exp(im\theta)$, only $m=\pm 1$ contribute to lowest order in our expansion of the linear theory].

Incorporating the linear analysis results (a)–(c) in (1), (2), (4), (9), and (10), we obtain the following model:

$$M\ddot{y} + 2M\zeta\Omega\dot{y} + M\Omega^2 y = \sum_i \bar{f}_i \cos(\theta_i + \beta_i), \qquad (13)$$

$$\dot{\theta}_i = \omega_i - c_i \ddot{y} \cos(\theta_i + \xi_i), \qquad (14)$$

where $\bar{f}_i \exp(i\beta_i) = f_i^{(1)}$, $c_i \exp(i\xi_i) = C_i$, and $(\bar{f}_i, \beta_i, c_i, \xi_i)$ are all real. In addition, by shifting the zero of θ_i, we can, without loss of generality, from now on take $\beta_i=0$ for all i.

We now offer some further discussion concerning assumptions that we have made in this section.

The assumption that the walker response $F_i(t)$ is linear in $\ddot{y}$: This assumption is clearly valid when one examines the linear stability of the pedestrian-bridge system. The system of equations that results from the assumption that $F_i(t)$ is linear in $\ddot{y}$ is, however, nonlinear, since (6) involves $\theta_i(t)$, which, by (1) drives the bridge response. In Sec. IV we will use (13) and (14) as a basis for investigating the nonlinear system response. We do this on the basis that (13) and (14) represent a lowest order model that satisfies the minimal requirements that it reproduces the correct form of the linear dynamics of the system, while at the same time depending simply on the bridge acceleration $\ddot{y}$. At small nonlinearity (e.g., as would apply when the marginal stability condition (see Sec. III) is just slightly exceeded), the most important correction to linearity of F_i in $\ddot{y}$ would be a term quadratic in $\ddot{y}$ whose main effect would be a nonlinear frequency shift of ω_i. By our subsequent use of (13) and (14), we are essentially neglecting this effect.

The assumption that (5) can be replaced by (6): That is, that $F_i(t)$ only depends on the current value of the phase, $\theta_i(t)$, rather than on its entire past history. If $\ddot{y}$ is assumed small, Eq. (14) implies that in Eq. (5) the past values of θ_i [i.e., $\theta_i(t'')$ for $t'' \le t$] can be approximated as

$$\theta_i(t'') \approx \theta_i(t) - \omega_i(t-t''), \qquad (15)$$

for

$$c_i|\ddot{y}|(t-t'') \ll \pi. \qquad (16)$$

In the linear phase of the instability, we formally have $\ddot{y} \to 0$, and condition (16) is, by definition, always satisfied. In the case of nonlinear evolution with $|\ddot{y}|$ small, Eq. (16) can be satisfied for a long time, $\omega_i(t-t') \gg \pi$. Further, it would seem reasonable that walker memory of the past history of θ_i would not extend over many walker periods. Thus for $\ddot{y}$ small (15) seems a reasonable approximation to insert for $\theta_i(t'')$ in (5). Furthermore, we note that (15) says that the past history of $\theta_i(t'')$ is completely determined by the current phase $\theta_i(t)$, hence leading to (6).

The assumption of linear bridge dynamics. In writing Eq. (1) we have taken the bridge motion to be linear. This is, of course, again valid for the treatment of the linear instability of the system. Our subsequent use of our model in the nonlinear regime (Sec. IV) essentially assumes that any nonlinearity in the bridge dynamics is weaker than the nonlinearity of the walker response due to the term $\ddot{y}\cos(\theta_i+\xi_i)$ in (14). Because of the narrowness of the resonance between the bridge and the pedestrians, the most likely bridge nonlinearity of potential interest is a nonlinear frequency shift of the bridge resonant frequency. While this effect could, in principal, be included within the present analysis framework (along with any similar frequency shift in ω_i), we will work here on the assumption that the most significant factor is the nonlinearity in the walker response term in (14).

Assumption that only lateral walker forces are important: Another presumption, implicit in Eq. (1) and our definition of $\hat{f}_i(t)$, is that the bridge response is driven by the *lateral* force exerted by the walkers. In particular, it is known [1,2,6,15] that the lateral force exerted by walkers is small compared to their back-to-front force, which in turn is small compared to their vertical force variation. Thus it might be asked whether our neglect of these forces is warranted. In the case of the linear analysis of the Millennium Bridge, focus on the lateral force is justified because the frequency of this force resonates with the bridge natural frequency Ω, and this is the observed frequency of the bridge response. In contrast, the frequency of the vertical and back-to-front walker forces are at a fundamental of $2\omega_i$ [i.e., a similar force is exerted every time a foot is put down (whether it be the left or the right foot)]. Thus nonlinearity would be necessary for these $2\omega_i$ forces to drive a bridge response at Ω. Even in the nonlinear regime, experiments indicate that vertical and back-to-front forcing are not significant for the bridge motion. For example, on page 26 of their paper, Dallard *et al.* [1] state that their data shows that "the lateral component of load was by far the most significant." Furthermore, experiments by McRobie *et al.* [15] on a large rig (which had only a lateral degree of freedom) were able to replicate the instability phenomenon observed on the Millennium Bridge, thus further supporting the assumption that lateral forces are the important effect.

III. LINEAR ANALYSIS

In this section we examine the linear stability problem for the bridge-pedestrian system using Eqs. (1), (2), (4), and (9)–(11). For this purpose, it is useful to assume that the i-dependence of walkers can be parametrized by a vector of walker parameters that we denote $\mathbf{p}_i$. Thus we write

$$\hat{\mathcal{L}}_i^{(n)}(\tau) = \hat{\mathcal{L}}^{(n)}(\tau,\mathbf{p}_i) \tag{17}$$

[where $\hat{\mathcal{L}}_i$ is defined in (8)]. The vector $\mathbf{p}_i$ also includes ω_i as a component and also characterizes the function $f_i(\theta)$, e.g., by including as components of $\mathbf{p}_i$ the Fourier coefficients $f_i^{(m)}$,

$$f_i(\theta) = \sum_m f_i^{(m)} \exp(im\theta), \tag{18}$$

where $f_i^{(0)}=0$ (otherwise, by Newton's third law, the bridge would exert an average lateral force on the walker, thus causing the walker to accelerate sideways).

We model the instantaneous state of the walkers on the bridge by a continuous distribution function $\rho(\theta,\mathbf{p},t)$ [12,13,16] such that

$$\int_0^{2\pi} \rho(\theta,\mathbf{p},t)d\theta = \hat{Q}(\mathbf{p}), \tag{19}$$

where $\rho(\theta,\mathbf{p},t)d\theta d\mathbf{p}$ is the fraction of walkers on the bridge whose phase angle is between θ and $\theta+d\theta$ and whose $\mathbf{p}$ vector lies in the parameter space volume $d\mathbf{p}$ centered at $\mathbf{p}$. The parameter distribution function $\hat{Q}(\mathbf{p})$ is independent of time (the parameters of an individual walker are assumed to not change with time). The description of the walker population via a continuous distribution function $\rho(\theta,\mathbf{p},t)$ is expected to be a good approximation insofar as the number of walkers is large $N \gg 1$. From (4) and (9)–(11), conservation of the number of walkers implies the following evolution equation for ρ:

$$\frac{\partial\rho}{\partial t} + \omega\frac{\partial\rho}{\partial\theta} - \frac{\partial}{\partial\theta}\left(\rho\sum_{n=-\infty}^{+\infty} e^{in\theta}\mathcal{L}_i^{(n)}[\ddot{y}]\right) = 0. \tag{20}$$

In addition, the right-hand side of (1) is

$$\sum_{i=1}^{N} \hat{f}_i(t) = N\int \rho(\theta,\mathbf{p},t)\sum_m f^{(m)}e^{im\theta}d\theta d\mathbf{p}. \tag{21}$$

A trivial solution to (1), (20), and (21) is

$$\rho_0 = \hat{Q}(\mathbf{p})/2\pi \tag{22}$$

and $y\equiv 0$. This corresponds to the case in which people on the bridge walk with their natural frequencies, and their effect on the bridge cancels because their phases are uniformly distributed in $0\leq\theta<2\pi$.

To study the stability of this state, we introduce a perturbation ρ_1 to (22),

$$\rho = \rho_0(\mathbf{p}) + \rho_1(\theta,\mathbf{p},t), \tag{23}$$

and linearize (20) for ρ_1 and y small. Assuming a time dependence $\exp(st)$ for the small quantities, we obtain from (20),

$$s\rho_1 + \omega\frac{\partial\rho_1}{\partial\theta} = is^2\rho_0 y\sum_{n=-\infty}^{+\infty} ne^{in\theta}C^{(n)}(s,\mathbf{p}), \tag{24}$$

where

$$C^{(n)}(s,\mathbf{p}) = \int_0^{\infty} e^{-s\tau}\hat{\mathcal{L}}^{(n)}(\tau,\mathbf{p})d\tau. \tag{25}$$

The solution of (24) is

$$\rho_1 = is^2\rho_0 y\sum_{n=-\infty}^{+\infty} n\frac{C^{(n)}(s,\mathbf{p})e^{in\theta}}{s+in\omega}. \tag{26}$$

Using (26) in (21) we obtain from (1)

$$s^2 + 2\zeta\Omega s + \Omega^2 = \frac{is^2N}{M}\int d\mathbf{p}\hat{Q}(\mathbf{p})\sum_{n=-\infty}^{+\infty} n\frac{C^{(n)}(s,\mathbf{p})f^{(-n)}}{s+in\omega}. \tag{27}$$

We now make use of the three facts mentioned in Sec. II concerning walkers on the Millennium Bridge (small bridge damping, $\zeta\ll 1$; small spread in the frequency distribution of walkers, $\sigma\ll\mu$; and near resonance between the average frequency μ of the walker population and the bridge, $|\mu-\Omega|\ll\Omega$). This suggests the following ordering for the quantities in (27):

$$s = -i\Omega + \epsilon\Omega s', \tag{28}$$

$$\zeta = \epsilon\zeta', \tag{29}$$

$$\omega = \Omega + \epsilon\Omega\omega', \tag{30}$$

$$\left(\frac{N}{2\Omega M}\right)C^{(n)}(-i\Omega,\mathbf{p})f^{(-n)} = \epsilon^2 B^{(n)}, \tag{31}$$

where $\epsilon \ll 1$ is an expansion parameter, and s', ζ', ω', and $B^{(n)}$ are all of order one. [We note that, since $y(t)$ is real, an equally valid form in place of (28) is $s=i\Omega+\epsilon\Omega s'$.] Note that, to lowest order in ϵ, the s dependence of $C^{(n)}(s,\mathbf{p})$ is ignorable, and $C^{(n)}(s,\mathbf{p})$ is replaced by $C^{(n)}(-i\Omega,\mathbf{p})$. This yields Eq. (12) with

$$C_i \equiv C^{(1)}(-i\Omega,\mathbf{p}_i),$$

justifying claim (b) of Sec. II. Inserting (28)–(31) in (27), the denominators of the integrand on the right-hand side of (27) yield $(s+in\omega)^{-1}=(i\Omega)[(n-1)+\epsilon(s'+in\omega')]^{-1}$. Thus, the $n \neq 1$ terms are smaller than the $n=1$ term by $O(\epsilon)$, and the $n=1$ term dominates, thus justifying claims (a) and (c) of Sec. II. [If $s=i\Omega+\epsilon\Omega s'$ replaces (28), then $n=-1$ dominates.] To lowest order we obtain

$$s' + \zeta' = -i\int d\mathbf{p}\frac{B^{(1)}\hat{Q}(\mathbf{p})}{\omega' - is'}. \tag{32}$$

Since $B^{(1)}$ and ω are parameters of individual oscillators we write the parameter vector as $\mathbf{p}=[\omega,\bar{B},\xi,\hat{\mathbf{p}}^{\dagger}]^{\dagger}$ where $\bar{B}e^{i\xi}=B^{(1)}$ and $\dagger$ denotes transpose. Thus we represent $d\mathbf{p}$ as $d\omega d\bar{B}d\xi d\hat{\mathbf{p}}$ and integrate over $\hat{\mathbf{p}}$ to obtain

$$s' + \zeta' = -i\int_0^{2\pi}d\xi\int_0^{\infty}d\bar{B}\int d\omega' Q(\omega',\bar{B},\xi)\frac{\bar{B}e^{i\xi}}{\omega' - is'}, \tag{33}$$

where $Q(\omega',\bar{B},\xi)$ is the joint distribution function of ω', $\bar{B}$, and ξ, and the contour of the ω' integration is taken in the causal sense [16], i.e., running from $\mathrm{Re}(\omega')=-\infty$ to $\mathrm{Re}(\omega')=+\infty$ passing *below* the pole $\omega'=is'$.

As an example, we consider the case where $\bar{B}$ and ξ are independent of ω',

$$Q(\omega',\bar{B},\xi) = P'(\omega')R(\bar{B},\xi), \tag{34}$$

where P' is the distribution function for ω', and R is the joint distribution function for $\bar{B}$ and ξ. Defining b' and ϕ by

$$\int_0^{2\pi}d\xi\int_0^{\infty}d\bar{B}R(\bar{B},\xi)\bar{B}e^{i\xi} = \frac{1}{4}b'e^{i\phi}, \tag{35}$$

Eq. (33) becomes

$$s' + \zeta' = -\frac{i}{4}b'e^{i\phi}\int\frac{P'(\omega')}{\omega' - is'}d\omega'. \tag{36}$$

We first analyze (36) by considering the case of marginal stability, $s' \to -i\omega_0'+0^+$, and take the integration contour in the complex ω' plane to run from $\mathrm{Re}(\omega')=-\infty$, $\mathrm{Im}(\omega')=0$, along the real ω' axis to the point $\mathrm{Re}(\omega')=\omega_0'-\Delta$, then to $\mathrm{Re}(\omega')=\omega_0'+\Delta$ along a radius Δ semicircle passing below the pole at $\omega'=\omega_0'$, and then from $\mathrm{Re}(\omega')=\omega_0'+\Delta$ to $\mathrm{Re}(\omega')=+\infty$ along the real ω' axis. Letting $\Delta \to 0^+$ this gives

$$-i\omega_0' + \zeta' = \frac{1}{4}b'e^{i\phi}\left(\pi P'(\omega_0') - iPV\int_{-\infty}^{+\infty}\frac{P'(\omega')}{\omega' - \omega_0'}d\omega\right), \tag{37}$$

and PV denotes a principal value integration. The distribution function for walker frequencies has previously been determined from data on a large population of walkers and fitted to a Gaussian distribution [17]. We therefore take

$$P'(\omega') = \frac{\exp[-(\omega' - \mu')/2\sigma'^2]}{\sqrt{2\pi}\sigma'}. \tag{38}$$

The simplest case is when the parameters μ' and ϕ are zero. In this case the imaginary part of (37) has a solution with $\omega_0'=0$. This follows since the principal value integral is zero because (for $\mu'=0$) $P'(\omega')$ is even about $\omega'=0$. The real part of (37) then yields a critical bridge damping $\zeta_{c0}'=(1/4)\sqrt{\pi/2}(b'/\sigma')$ such that, if $\zeta' < \zeta_{c0}'$, the bridge-pedestrian system is unstable. (The subscript zero on ζ_{c0}' denotes the case $\mu'=\phi=0$.) Introducing the more physical variables, $\zeta=\epsilon\zeta', \sigma/\Omega=\epsilon\sigma'$, $b=\epsilon^2 b'$, we obtain the instability condition

$$\zeta < \zeta_{c0} = \sqrt{\frac{\pi}{2}}\frac{b\Omega}{4\sigma}. \tag{39}$$

For $P'(\omega')$ even and monotonically decreasing away from $\omega'=0$ [as in Eq. (38) with $\mu'=0$] and $\phi=0$, the relevant root s' of (36) is real for all ζ, with $s'>0$ corresponding to instability and $s'<0$ corresponding to stability. To determine s' we now take the integration path to run from $\mathrm{Re}(\omega')=-\infty$ to $\mathrm{Re}(\omega')=+\infty$ along $\mathrm{Im}(\omega')=is'$, with a small semicircular indentation below the pole at $\omega'=is'$. Using this path, we introduce the change of variables

$$x = (\omega' - is')/\sigma'$$

in terms of which (36) becomes

$$s' + \zeta' = -\frac{ib'}{4}\int\frac{dx}{x}P'(\sigma'x + is'). \tag{40}$$

Inserting the Gaussian form (38) with $\mu'=0$ into (40) yields

$$s' + \zeta' = \zeta_{c0}'\exp(y^2/2)\left(1 - \sqrt{\frac{2}{\pi}}I(y)\right), \tag{41}$$

where $y=s'/\sigma'$ and $I(y)$ denotes the integral,

$$I(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\exp(-z^2/2)}{z}\sin(yz)dz, \tag{42}$$

which can be shown to be

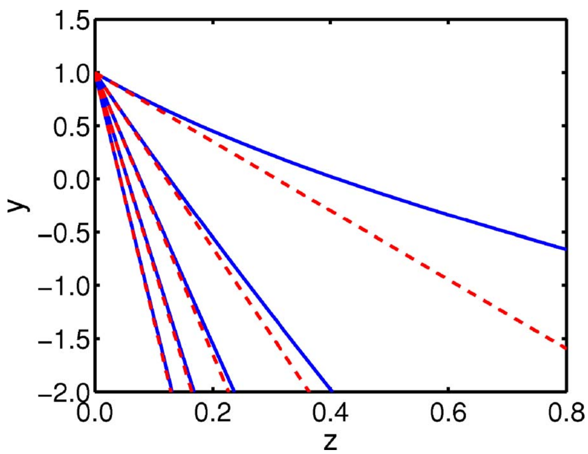

FIG. 1. (Color online) Normalized growth rate $y=\gamma/\sigma$ vs the normalized bridge damping $z=\zeta/\zeta_{c0}$ for different values of $a=\sqrt{8\pi}\sigma^2/(\Omega^2 b)$ as calculated from Eq. (44). The full blue lines are the actual results from (44), the dashed red ones the corresponding linear approximations from Eq. (45). The values for a increase from the lower left to upper right and are a=0.5, 1.5, 2.5, 3.5, and 4.5.

$$I(y) = \int_0^y \exp(-z^2/2)dz = \sqrt{\pi/2}\mathrm{erf}(y/\sqrt{2}), \tag{43}$$

where erf(u) is the error function of u. Thus (41) becomes

$$\zeta/\zeta_{c0} = \exp(y^2/2)[1-\mathrm{erf}(y/\sqrt{2})] - \sqrt{2/\pi}[\Omega^2 b/(4\sigma^2)]^{-1} y, \tag{44}$$

where $y=\gamma/\sigma$, and $\gamma \equiv \mathrm{Re}(s)$. Equation (44) gives an explicit expression for ζ as a function of the exponential growth or damping rate γ ($\gamma>0$ for growth). Figure 1 shows the growth rate as a function of the damping plotted as γ/σ versus ζ/ζ_{c0} for several values of $\Omega^2 b/(4\sigma^2)$. Expanding (44) for $|\zeta/\zeta_{c0}-1|$ small we obtain

$$\gamma = \frac{\zeta_{c0}-\zeta}{1+[\Omega^2 b/(4\sigma^2)]}\Omega. \tag{45}$$

Equation (39) applies for $\mu'=\phi=0$. We now consider how (39) is modified for nonzero but small μ' and ϕ. In particular, we consider the distribution function (38) and do a perturbation expansion of (37) about $\mu'=\phi=0$ considering $\omega_0'/\sigma' \sim \mu'/\sigma' \sim \phi \ll 1$. Writing

$$P'(\omega') = P'[(\omega'+\mu'-\omega_0') - (\mu'-\omega_0')]$$

and expanding for small $(\mu'-\omega_0')$, we have

$$P'(\omega') \cong P'(\omega'+\mu'-\omega_0')[1+(\mu'-\omega_0')(\omega'-\omega_0')/\sigma^2]. \tag{46}$$

Inserting (46) into the principal value integral yields

$$PV\int_{-\infty}^{+\infty} \frac{P'(\omega')}{\omega'-\omega_0'} d\omega' \cong -\frac{\omega_0'-\mu'}{\sigma'^2}. \tag{47}$$

We also have $P'(\omega_0') \cong \left[1-\frac{(\omega_0'-\mu')^2}{2(\sigma')^2}\right]\frac{1}{\sqrt{2\pi}\sigma'}$, $e^{i\phi} \cong 1+i\phi-\frac{1}{2}\phi^2$. Using these and (47) in (37), and defining the normalized frequency shift between ω_0 and μ by $\delta=(\omega_0-\mu)/\sigma$, we obtain the critical bridge damping value $\zeta_c(\delta,\phi)$ for $|(\mu-\Omega)/\sigma| \sim \phi \ll 1$,

$$[(\zeta_c(\delta,\phi)/\zeta_{c0}) - 1] \cong -\frac{1}{2}\left(\delta^2 + 2\sqrt{\frac{2}{\pi}}\phi\delta + \phi^2\right), \tag{48}$$

where

$$\delta = -\left[\left(\frac{\mu-\Omega}{\sigma}\right) + \zeta_{c0}\phi\right] \Big/ \left(1+\sqrt{\frac{2}{\pi}}\zeta_{c0}\right). \tag{49}$$

Since the coefficient of the $\phi\delta$ term in (48) is less than 2, the right-hand side of (48) is always negative if $(\delta,\phi) \neq (0,0)$, making ζ_c smaller than ζ_{c0}. This indicates that, within the range $|(\mu-\Omega)/\sigma| \sim \phi \ll 1$, the situation, $\mu-\Omega$, $\phi=0$, represents a worst-case-scenario in that it is the most unstable. Our numerical results of Sec. IV B indicate that this conclusion persists when $|(\mu-\Omega)/\sigma|$ and ϕ are not small.

We now briefly consider the effect of nondeterministic, randomlike inputs modifying the phase of walkers. Such inputs might include social interactions with nearby walkers, stumbles, erratic walker mood changes, etc. A simple means of modeling these effects is as a zero-mean additive random increment to the right-hand side of Eq. (4). Denoting this increment by $\eta_i(t)$ and taking it to be approximately delta correlated in time, we have

$$\langle \eta_i(t)\eta_i(t')\rangle = 2D_{\theta\theta}\delta(t-t'), \tag{50}$$

where the angle brackets indicate an ensemble average, and $D_{\theta\theta}$ is a parameter that characterizes the strength of the random inputs. Incorporating (50) via a Langevin-type analysis results in an additional term $D_{\theta\theta}\partial^2\rho/\partial\theta^2$ appearing on the right-hand side of (20). Thus $D_{\theta\theta}$ can be interpreted as a phase diffusion coefficient. As a result, the denominator of the summand appearing in (27) becomes $[(s+n^2 D_{\theta\theta})+in\omega]$ and the denominator in the integrand appearing in (36) becomes $[\omega'-i(s+D_{\theta\theta})]$. Analysis of (36) with this replacement shows that ζ_{c0} is decreased by the presence of $D_{\theta\theta}$; i.e., such random inputs are stabilizing.

IV. NONLINEAR BEHAVIOR

In this section we adopt (13) and (14) as a model for the dynamics describing the nonlinear evolution of the bridge-pedestrian system. This model satisfies the minimal requirements that it depends simply on the bridge acceleration $\ddot{y}$ and that it reproduces the correct form of the linear dynamics of the system. On the other hand, this model is not necessarily expected to yield accurate quantitative agreement with experiments in the nonlinear regime. Nevertheless, the model might be reasonably anticipated to yield qualitative features of the observed motion, as well as rough quantitative agreement with the observations. Indeed, this will be shown to be the case. A further point to note is that our current state of knowledge concerning the response of walkers to external forces is very meager, and does not justify a more elaborate model. Thus we view results from (13) and (14) as a providing guide to future studies rather than as a definitive description of the system's nonlinear behavior. In this spirit, and in order to simplify our analysis, we henceforth neglect the variation of the parameters ξ_i, $\bar{f}_i$, and c_i from walker to walker; that is, we set

$$\bar{f}_i = f, \quad c_i = c, \quad \xi_i = \phi$$

for all i. Since we do not know the distribution of these parameters, this choice is as good as any, and it is hoped that qualitative results will not be sensitive to it. Formulations like this have often been used for mathematically similar problems that arise in other fields (e.g., biological rhythms [19,20], such as the synchronized flashing of fireflies), and have been found in those cases to be very useful. As previously mentioned, we use a Gaussian model for the distribution function of the walker frequencies,

$$P(\omega) = (\sqrt{2\pi}\sigma)^{-1}\exp[-(\omega-\mu)^2/(2\sigma^2)]. \tag{51}$$

For the purposes of further analysis it is useful to set the equations into a dimensionless form; the new dimensionless quantities will be distinguished by a tilde over the symbols. We rescale time $t=\tilde{t}/\Omega$, amplitude $y=Nf/(M\Omega^2)\tilde{y}$, and frequencies $\omega_i=\tilde{\omega}_i\Omega$, $\mu=\tilde{\mu}\Omega$, $\sigma=\tilde{\sigma}\Omega$, so that

$$\ddot{\tilde{y}} + 2\zeta\dot{\tilde{y}} + \tilde{y} = \langle\cos\theta\rangle, \tag{52}$$

$$\dot{\theta}_i = \tilde{\omega}_i - b\ddot{\tilde{y}}\cos(\theta_i+\phi), \tag{53}$$

and the distribution function for $\tilde{\omega}$ is

$$\tilde{P}(\tilde{\omega}) = (\sqrt{2\pi}\tilde{\sigma})^{-1}\exp[-(\tilde{\omega}-\tilde{\mu})/(2\tilde{\sigma}^2)].$$

Here $\langle\cos\theta\rangle$ denotes the average of $\cos\theta_i$ over all the walkers ($i=1,2,\ldots,N$), and the dots now denote derivatives with respect to dimensionless time $\tilde{t}$. Except for the damping, all bridge and individual walker characteristics are then condensed into two dimensionless parameters, ϕ and

$$b = Nf/(\Omega\tau M g_0), \tag{54}$$

which we call the coupling strength.

A. Analysis

It turns out that for the case $\phi=0$ and $\tilde{\mu}=1$, analytical results can be obtained. In this section we restrict consideration to this case. In Sec. IV B, where we study the model, (52), (53), numerically, some results for $\phi\neq 0$, $\tilde{\mu}\neq 1$ are given. Further consideration of $\phi\neq 0$, $\tilde{\mu}\neq 1$, as well as more extensive simulations and the more technically involved analytical discussion are left for future investigation. (Note that $\tilde{\mu}=1$ implies that the peak of the Gaussian walker frequency distribution function occurs at the bridge resonant frequency.)

The analysis proceeds in a manner similar to that originally followed by Kuramoto [12,13]. We seek a solution of (52) and (53) in which the bridge oscillates at its resonant frequency, i.e., $\tilde{y}(\tilde{t})=\tilde{A}\sin\tilde{t}$. With this form for $\tilde{y}(\tilde{t})$ we have

$$-\ddot{\tilde{y}}\cos\theta_i = \frac{1}{2}\tilde{A}[\sin(\tilde{t}-\theta_i)+\sin(\tilde{t}+\theta_i)]. \tag{55}$$

Anticipating that $|\dot{\theta}_i-1|$ is small, the second term varies rapidly compared to the first. We thus neglect its effect on the right-hand side of (53), which then becomes

$$\dot{\theta}_i \cong \tilde{\omega}_i + \tfrac{1}{2}\tilde{A}b\sin(\tilde{t}-\theta_i). \tag{56}$$

From (56) those pedestrians whose natural stepping frequencies are close enough to the bridge resonance will become synchronized with the bridge oscillation, i.e., $\dot{\theta}_i=1$ for these pedestrians and $\theta_i(\tilde{t})=\tilde{t}+\theta_{i0}$, where

$$\sin\theta_{i0} = 2\frac{\tilde{\omega}_i-1}{b\tilde{A}} \tag{57}$$

and $|\tilde{\omega}_i-1|<\frac{1}{2}b\tilde{A}$ for the synchronized pedestrians.

With the bridge responding at its eigenfrequency, the acceleration and harmonic restoring force in (52) cancel and the force from the pedestrians is balanced by the frictional force,

$$2\zeta\tilde{A}\cos\tilde{t} = \langle\cos\theta\rangle. \tag{58}$$

It may be shown (e.g., see Refs. [12,13]) that only the synchronized walkers satisfying $|\tilde{\omega}_i-1|<\frac{1}{2}b\tilde{A}$ make a nonzero contribution to $\langle\cos\theta\rangle$. For the synchronized walkers we have

$$\langle\cos\theta\rangle = \langle\cos(\tilde{t}+\theta_{i0})\rangle = \cos\tilde{t}\langle\cos\theta_{i0}\rangle - \sin\tilde{t}\langle\sin\theta_{i0}\rangle, \tag{59}$$

where θ_{i0} is given by (57). For $\tilde{\omega}_i$ distributed with even symmetry about $\tilde{\omega}_i=1$ [as in (51) with $\tilde{\mu}=1$], Eq. (57) yields $\langle\sin\theta_{i0}\rangle=0$. Thus (58) can be satisfied and yields

$$2\zeta\tilde{A} = \langle\cos\theta_{i0}\rangle. \tag{60}$$

This relation is the origin of the force-velocity relation determined by Dallard *et al.* [1–3].

As in Sec. III we consider the number of walkers on the bridge to be large ($N\gg 1$) and adopt a continuum description. Using (57) the average needed for (60) can then be expressed

$$\langle\cos\theta_{i0}\rangle = \frac{b\tilde{A}}{2}\int_{-1}^{+1} dz\tilde{P}\left(1+\frac{1}{2}b\tilde{A}z\right)\sqrt{1-z^2}, \tag{61}$$

where the contribution from the unsynchronized pedestrians has been taken to be zero for $\tilde{P}(\tilde{\omega})$ symmetric about $\tilde{\omega}=1$ (see standard treatments of the Kuramoto problem [12,13]).

When combined, Eqs. (60) and (61) give

$$\frac{\zeta}{b}\tilde{A} = \tilde{A}\frac{1}{8\tilde{\sigma}\sqrt{2\pi}}e^{-\eta^2}[I_0(\eta^2)+I_1(\eta^2)], \tag{62}$$

where $\eta^2=(b^2\tilde{A}^2/8\tilde{\sigma}^2)$ and $I_n(\eta^2)$ is the nth order modified Bessel function of argument η^2, i.e., $I_n(\eta^2) = \pi^{-1}\int_0^\pi\cos(n\theta)\exp(\eta^2\cos\theta)d\theta$. A trivial solution to (62) is

$\tilde{A}=0$, corresponding to the state when the bridge is at rest and the pedestrians are not synchronized. If ζ/b is small enough, (62) has a nontrivial solution, $\tilde{A}>0$, corresponding to oscillation of the bridge. The requirement on ζ/b for the presence of synchronization is the same as the condition for linear instability given by Eq. (39). Since b in(39) is related to the number of pedestrians by (54), we can express this in the form of a critical number N_c of people above which oscillations set in,

$$N_c = 4\zeta M\Omega\left(\frac{2\tau g_0}{\pi\tilde{P}(1)f}\right), \tag{63}$$

where $\tilde{P}(1)=(\tilde{\sigma}\sqrt{2\pi})^{-1}$. Comparing this expression with that found by Dallard *et al.* [1–3],

$$N_{c,\mathrm{Dallard}} = 4\zeta M\Omega/k \tag{64}$$

we see that the constant k of dimensions (force/velocity) can now be expressed as a function of the parameters of the model,

$$k = \frac{\pi\tilde{P}(1)f}{2\tau g_0}. \tag{65}$$

B. Numerical studies

The discussion in the preceding section suggests a minimal model for the behavior of pedestrians on the bridge, Eqs. (52) and (53) (dropping the tildes for the dimensionless quantities for the remainder of this section)

$$\ddot{y} + 2\zeta\dot{y} + y = \langle\cos\theta\rangle, \tag{66}$$

$$\dot{\theta}_i = \omega_i - b\ddot{y}\cos(\theta_i - \phi). \tag{67}$$

The stepping frequencies ω_i of the pedestrians are drawn from a Gaussian distribution

$$P(\omega) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(\omega-\mu)/(2\sigma^2)}. \tag{68}$$

The parameters for a numerical simulation thus are

ζ, damping rate of the bridge motion, normalized to the bridge resonant frequency;

b, coupling strength between bridge and pedestrians;

ϕ, phase delay in the phase oscillator;

σ, variance of the stepping frequency;

μ, peak position of the walker frequency distribution; normalized to the bridge resonant frequency;

N, number of pedestrians.

For the other parameters we assume that the phase delays ϕ and the coupling strengths b are the same for all pedestrians, and that the N frequencies are randomly drawn from the Gaussian distribution. The width is fixed at value $\sigma=0.086$, as measured in Ref. [14].

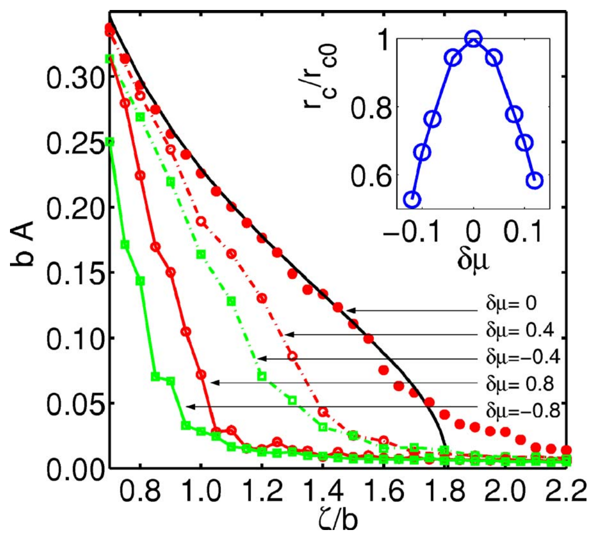

FIG. 2. (Color online) The dimensionless amplitude bA of the bridge motion vs the ratio $r=\zeta/b$. For the simulations (circles) an average over 20 realizations of the frequency distribution for 200 oscillators and uniformly distributed initial conditions were averaged. Each realization was run for 300 periods and the amplitude was determined from the square root of the mean of $\tilde{y}^2+\dot{\tilde{y}}^2$ over the last 100 periods. The black curve represents the theoretical prediction (62). The critical value here is $r_{c0}=1.83$, determined with a width $\sigma=0.086$. The other four curves are obtained for a mismatch $\delta\mu=\tilde{\mu}-1$ in frequency, $\delta\mu=\pm0.04$ and $\delta\mu=\pm0.08$ between eigenfrequency of the bridge and preferred frequency of the people. We use this data for $bA\geq0.5$ to extrapolate r to $bA=0$, thus obtaining estimates of the critical value r_c at oscillation onset for $\delta\mu\neq0$. These results are plotted in the inset as r_c/r_{c0} versus $\delta\mu$, where r_{c0} is the value for r_c at $\delta\mu=0$.

For our first set of simulations, we consider the case $\phi=0$, and a group of 200 walkers. Figure 2 shows the time asymptotic amplitude of the bridge vs the linear stability parameter $r=\zeta/b$. The critical value is $r_c=\pi P(1)/8=1.83$, so that one expects a zero amplitude for $r>r_c$, rising like a

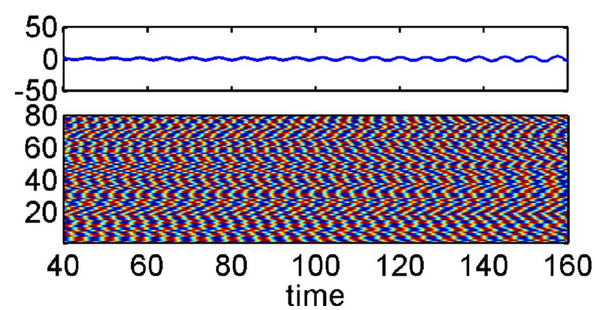

FIG. 3. (Color online) Synchronization in a group of 80 walkers represented in a phase color plot. The upper frame shows the bridge oscillation, the lower one color coding of the phases of 80 walkers. Alternating steps with the left and right foot appear as red and blue, or light and dark, respectively. For a coupling parameter of $b=0.01$ the walkers do not synchronize and the bridge wobbles with small amplitude.

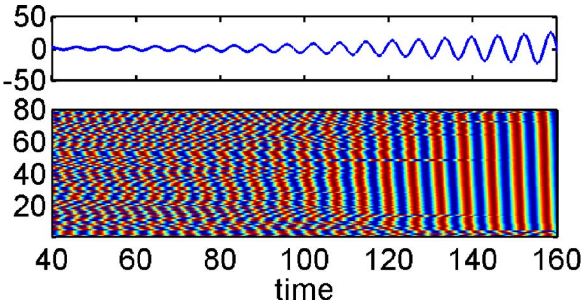

FIG. 4. (Color online) Same as Fig. 3, but now for $b=0.02$, where synchronization is observed.

square root just below $r<r_c$, i.e., $bA(r)\sim\sqrt{r_c-r}$, and going over to the nonlinear behavior contained in Eq. (62). The nonzero values for large r are a finite size effect due to a finite N; see Ref. [21] for a discussion of effects from finite numbers of oscillators.

The inset in Fig. 2 shows the dependence of the critical value on the mismatch $\delta\mu=\mu-1$ between the original maximum of the stepping frequency and the eigenfrequency of the bridge. When the center of the frequency distribution moves away from perfect resonance, $\delta\mu\neq 0$, the critical threshold r_c drops quickly over a range of about one standard deviation σ in the frequency distribution. This means that weaker damping or more people are required to trigger the onset of synchronization, as predicted by Eq. (48).

The onset of oscillations can be visualized in phase plots for the oscillators. We take the cosine of the phase and color code its range from -1 to $+1$ from red to blue. A full period hence corresponds to a change from red to blue and back to red. Synchronized walkers will then have the same colors.

Figure 3 shows the dynamics of 100 oscillators for values of the parameter r well below onset. The bridge shows a weak residual oscillation and the pedestrians walk essentially with their unperturbed frequency. Correlations in motion of the pedestrians are due to accidental alignments in phase and are transient. As the coupling is increased, synchronization sets in, starting in the center with oscillators whose frequencies coincide with the bridge eigenmode (see Fig. 4). For even larger coupling synchronization sets in more quickly (Fig. 5).

A magnification of Fig. 5 in Fig. 6 shows that the time it takes for a walker to synchronize (if it synchronizes at all) varies considerably from walker to walker, even if they are near in frequency.

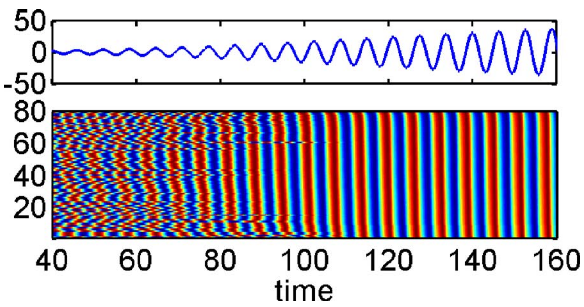

FIG. 5. (Color online) Same as Fig. 3, but now for $b=0.03$, where synchronization sets in quickly.

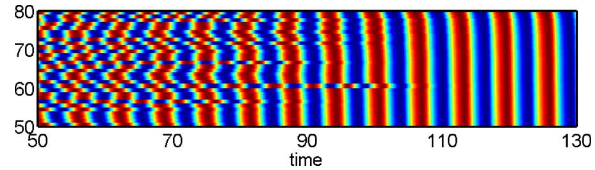

FIG. 6. (Color online) Magnification of a part of Fig. 5, showing the variation in times at which walkers are entrained.

The short time behavior of the numerical model shows that, for the specific parameters used to calculate Fig. 7(a), synchronization sets in after about 200 periods. Strikingly, Fig. 7(b) shows that data points for force during one period vs the maximal velocity fall near a straight line through the origin, similar to the relation, force proportional to bridge velocity, identified by Dallard *et al.* [2,3]. In addition, the data also show the clockwise loops noted in presynchronization events in the field studies (compare Fig. 17 of Ref. [2]).

V. ESTIMATING PARAMETERS

All but one parameter of the model can be extracted from data and observations: the exception is the parameter b controlling the rate with which pedestrians adjust their stepping phase to the bridge oscillation. However, as we will see, a plausible estimate gives results for the critical number of people needed to start the oscillations in good agreement with the actual values for the Millennium Bridge.

The parameter b incorporates the response time scale and sensitivity of the walkers to lateral forces. Studies of the behavior of people on moving structures [1,2,10,17] convincingly show that walkers are quite sensitive to small lateral accelerations of a fraction of a percent of the earth acceleration. People show measurable response to accelerations by adjusting their walking at lateral acceleration levels be-

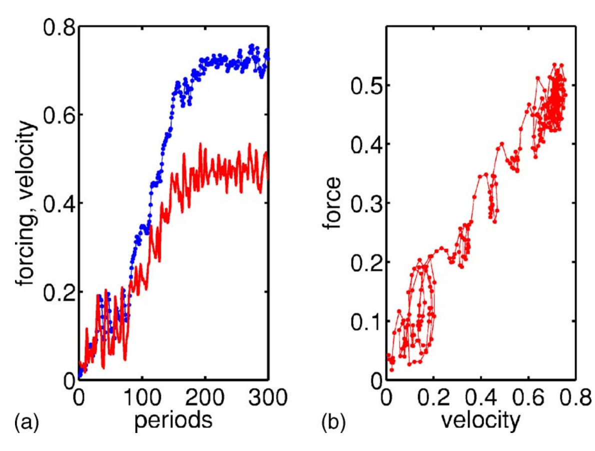

FIG. 7. (Color online) Time dependence of the synchronization event. (a) A time trace of the maximal amplitude of the force (continuous, red) and the maximal velocity of the bridge (times 0.1) (dotted, blue) during one period shows that the resonance develops on a time scale of about 200 periods. Here $\zeta/b=1.65$. (b) A plot of the force vs velocity of the bridge shows their approximate proportionality and the clockwise loops during the development stage, similar to the loops seen in the studies of Dallard *et al.* [2].

tween 0.2 m/s^2 and 0.5 m/s^2. As we will need a reference value for the pedestrian response below, we use $g_0 = 0.3\ m/s^2$, as established by Nakamura [17]. The parameter c also contains a time scale for the adjustment of stepping that we denote τ. We estimate τ to be of the order of the stepping period, which is approximately one second for typical walkers. Thus we model the constant b by

$$b = (g_0 \tau)^{-1}, \tag{69}$$

where τ is expected to be of the order of one second.

The remaining parameters can be obtained from the data on walker forces on the bridge. With $f=25N$ (Ref. [20]), $\Omega = 2\pi$ (Refs. [1–3]), $\tilde{\sigma}=0.09$ (Ref. [14]), and $g_0=0.3\ m/s^2$, Eq. (65) yields $k=(580/\tau)N$ s/m. This is to be compared with the value established by Dallard *et al.*, $k \approx 300N$ s/m, suggesting $\tau=1.9$ s, confirming that τ is of the order of a second, as originally supposed. From (64) the calculated value of k will decrease if pedestrians adjust their step placement to the vibrations less quickly, if the frequency distribution is wider than estimated in Ref. [14], or if the forcing amplitude is reduced. It will move up if the acceleration response constant g_0 is reduced.

VI. CLOSING REMARKS

The model proposed in Ref. [11] and here is, we believe, a minimal model that describes the onset of synchronized walking and lateral oscillations on bridges. By tracing the consequences of terms in a more general class of models we have shown here, that at least for the onset of the synchronization, how the more general model can be reduced to the basic ones used in Ref. [11] and in the numerical simulations in Sec. IV B.

The main modeling assumption and free parameter enters in the coupling between the phase dynamics of the walking and the bridge oscillation. The specific coupling used can be tested experimentally, e.g., by monitoring the walking of people and the bridge oscillation, as in the studies of Refs. [17,18]. It is reassuring that a consistent description of the experimental observations can be achieved with a plausible estimate for this parameter.

The dependence of the synchronization on the frequency distribution can be verified, for instance, by externally stimulating the walkers with the help of a metronome to move in synchrony (as in Ref. [22]): this will narrow their frequency distribution and hence bring down the critical number for inducing a transition.

ACKNOWLEDGMENTS

One of the authors (B.E.) would like to thank the Burgers Program at the University of Maryland, College Park, and in particular D.P. Lathrop for hospitality. The authors thank the German Research Foundation, ONR (Physics) and NSF (Grants Nos. PHY 045624 and DMS 0412757) for their support.

[1] P. Dallard, A. J. Fitzpatrick, A. Flint, S. LeBourva, A. Low, R. M. Ridsdill Smith, and M. Willford, Struct. Eng. **79**, 17 (2001).
[2] P. Dallard, A. J. Fitzpatrick, A. Flint, A. Low, R. M. Ridsdill Smith, M. Willford, and M. Roche, J. Bridge Eng. **6**, 412 (2001).
[3] A. J. Fitzpatrick, P. Dallard, S. Le Bourva, A. Low, R. M. Ridsdill Smith, and M. Willford, *Linking London: The Millennium Bridge* (The Royal Academy of Engineering, London, 2001).
[4] K. Yusuf Billah and R. H. Scanlan, Am. J. Phys. **59**, 118 (1991).
[5] T. M. Roberts, Bridge Eng. **156**, 155 (2003).
[6] D. E. Newland, J. Mech. Eng. Sci. **218**, 477 (2004).
[7] S. Nakamura, J. Struct. Eng. **130**, 32 (2004).
[8] B. Josephson, Letter to the Guardian, June 14, 2000. Reprinted in S. Strogatz, *Sync* (Hyperion, New York, 2003).
[9] H. Haken, *Synergetics, Introduction and Advanced Topics* (Springer, Berlin, 2004).
[10] S. Zivanović, A. Pavic, and P. Reynolds, J. Sound Vib. **279**, 1 (2005).
[11] S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, Nature (London) **438**, 43 (2005).
[12] Y. Kuramoto, *Chemical Oscillation, Waves and Turbulence* (Springer, Berlin, 1984).
[13] E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge University Press, Cambridge, 1998), pp. 236–244.
[14] Y. Matsumoto, S. Sato, T. Nishioka, and H. Shiojiri, Trans. Jpn. Soc. Civ. Eng. **4**, 50 (1972).
[15] A. McRobie, G. Morgenthal, J. Lasenby, and M. Ringer, Bridge Eng. **156**, 71 (2003).
[16] S. H. Strogatz, R. E. Mirollo, and P. C. Matthews, Phys. Rev. Lett. **68**, 2730 (1992).
[17] S. Nakamura, Struct. Eng. **81**, 22 (2003).
[18] Y. Fujino, B. M. Pacheco, S. Nakamura, and P. Warnitchai, Earthquake Eng. Struct. Dyn. **22**, 741 (1993).
[19] A. T. Winfree, *The Geometry of Biological Time*, 2nd ed. (Springer, New York, 2001).
[20] L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rhythm of Life* (Princeton University Press, Princeton, NJ, 1988).
[21] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005).
[22] S. Yao, J. R. Wright, A. Pavic, and P. Reynolds, Can. J. Civ. Eng. **31**, 1109 (2004).